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Hertz potentials in uniaxially anisotropic regions

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Abstract
Hertz potentials are used as an alternative to Fresnel's equation of wave normals to analyse harmonic plane wave propagation in uniaxially anisotropic media. Wave vector and amplitudes of ordinary and extraordinary waves are explicitly given. Refraction of a TM field at the plane face of a uniaxial medium is discussed and it is shown that in this particular situation, the refracted wave is identified with the extraordinary wave. Hertz potentials are also a powerful tool to tackle the same problems when harmonic plane waves are changed into Gaussian beams.

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1. Introduction

As stated in [1] 'to understand electromagnetic propagation in a medium whose characteristics are a function of the propagation direction, it is useful to consider the simplest type of anisotropy: the uni-axial'. For harmonic plane waves, the Fresnel's equation of wave normals is the conventional method for analysing propagation in anisotropic media [2–4]. We depart here from this technique, working instead with Hertz potentials [4, 5]: they explicitly give the wave vectors and the amplitudes of propagating fields. This makes possible a complete description of refraction in a uniaxial medium. Hertz potentials are shown to tackle the same problems efficiently with Gaussian beams instead of harmonic plane waves.

We start with the Maxwell equations; $\exp(\mathrm{i}\omega t)$ is implicit,

$$\nabla \wedge \mathbf{H} - \mathrm{i}\omega \mathbf{D} = 0 \qquad (1a)$$

$$\nabla \wedge \mathbf{E} + \mathrm{i}\omega \mathbf{B} = 0 \qquad (1b)$$

in a medium with the constitutive relations between the fields **E**, **D**, **H**, **B**

$$\mathbf{B} = \mu \mathbf{H}, \qquad D_x = \varepsilon E_x, \qquad D_y = \varepsilon E_y, \qquad D_z = \eta E_z, \qquad (2a)$$

and we use the notations

$$n^2 = \varepsilon\mu, \qquad m^2 = \eta\mu. \tag{2b}$$

We introduce the Hertz potentials $\mathbf{\Pi}(\mathbf{x})$ satisfying equation (1*b*)

$$\mathbf{H}(\mathbf{x}) = \mathrm{i}\omega\nabla\wedge\mathbf{\Pi}(\mathbf{x}), \qquad \mathbf{E}(\mathbf{x}) = \mu\omega^2\mathbf{\Pi}(\mathbf{x}). \tag{3}$$

They differ from the usual Hertz vectors [4, 5] in which $\mathbf{E} = \mu\omega^2\mathbf{\Pi} + \nabla\nabla\cdot\mathbf{\Pi}$.

Substituting (3) into (1*a*) gives, since $\nabla\wedge\nabla\wedge\mathbf{\Pi}(\mathbf{x}) = \nabla\nabla\cdot\mathbf{\Pi}(\mathbf{x}) - \Delta\mathbf{\Pi}(\mathbf{x})$,

$$\nabla\nabla\cdot\mathbf{\Pi}(\mathbf{x}) - \Delta\mathbf{\Pi}(\mathbf{x}) - \mathbf{D}(\mathbf{x}) = 0 \tag{4}$$

that is using (2*a*) and (2*b*), the argument $\mathbf{x}$ being deleted when there is no risk of confusion,

$$\begin{aligned}(\Delta+\omega^2n^2)\Pi_x &= \partial_x(\partial_x\Pi_x+\partial_y\Pi_y+\partial_z\Pi_z)\\ (\Delta+\omega^2n^2)\Pi_y &= \partial_y(\partial_x\Pi_x+\partial_y\Pi_y+\partial_z\Pi_z)\\ (\Delta+\omega^2m^2)\Pi_z &= \partial_z(\partial_x\Pi_x+\partial_y\Pi_y+\partial_z\Pi_z)\end{aligned} \tag{5a}$$

or

$$\begin{aligned}\left(\partial_y^2+\partial_z^2+\omega^2n^2\right)\Pi_x &= \partial_x(\partial_y\Pi_y+\partial_z\Pi_z)\\ \left(\partial_z^2+\partial_x^2+\omega^2n^2\right)\Pi_y &= \partial_y(\partial_z\Pi_z+\partial_x\Pi_x)\\ \left(\partial_x^2+\partial_y^2+\omega^2m^2\right)\Pi_z &= \partial_z(\partial_x\Pi_x+\partial_y\Pi_y).\end{aligned} \tag{5b}$$

Since we are interested in harmonic plane wave propagation, we look for the solutions of (5*b*) in the form

$$\mathbf{\Pi}(\mathbf{x}) = \mathbf{\Omega}\exp[\mathrm{i}(\alpha x+\beta y+\gamma z)] \tag{6}$$

in which $\mathbf{\Omega}$ is a constant vector depending, of course, on α, β, γ.

Taking (6) into account, we get from (5*b*) the homogeneous set of equations

$$\begin{aligned}&(n^2\omega^2-\beta^2-\gamma^2)\Omega_x+\alpha(\beta\Omega_y+\gamma\Omega_z)=0\\ &(n^2\omega^2-\alpha^2-\gamma^2)\Omega_y+\beta(\alpha\Omega_x+\gamma\Omega_z)=0\\ &(m^2\omega^2-\beta^2-\alpha^2)\Omega_z+\gamma(\alpha\Omega_x+\beta\Omega_y)=0\end{aligned} \tag{7a}$$

that we write as

$$\begin{vmatrix} n^2\omega^2-\beta^2-\gamma^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & n^2\omega^2-\alpha^2-\gamma^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & m^2\omega^2-\beta^2-\alpha^2 \end{vmatrix} \begin{vmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{vmatrix} = 0. \tag{7b}$$

Making null the determinant of this system gives the directions of propagation.

2. Harmonic plane wave propagation

2.1. Hertz potentials

Multiplying by $-\gamma/\beta$ the second column of the determinant in (7*b*) and summing with the third column gives the equation

$$\begin{vmatrix} a+\alpha^2 & \alpha\beta & 0 \\ \alpha\beta & a+\beta^2 & -a\gamma/\beta \\ \alpha\gamma & \beta\gamma & b \end{vmatrix} = 0 \tag{8a}$$

in which

$$a = n^2\omega^2-\alpha^2-\beta^2-\gamma^2, \qquad b = m^2\omega^2-\alpha^2-\beta^2-\gamma^2. \tag{8b}$$

Multiplying the second line by $-\gamma/\beta$ and summing with the third line, we get

$$\begin{vmatrix} a+\alpha^2 & \alpha\beta & 0 \\ \alpha\beta & a+\beta^2 & -a\gamma/\beta \\ 0 & -a\gamma/\beta & b+a\gamma^2/\beta^2 \end{vmatrix} = 0. \tag{9}$$

Expanding this last determinant gives

$$(a+\alpha^2)[(a+\beta^2)(b+a\gamma^2/\beta^2) - a\gamma^2/\beta^2] - \alpha^2(b\beta^2 + a\gamma^2) = 0 \tag{10a}$$

that reduces to

$$a[ab + b(\alpha^2+\beta^2) + a\gamma^2] = 0 \tag{10b}$$

with the two solutions

$$a = 0, \qquad ab + b(\alpha^2+\beta^2) + a\gamma^2 = 0 \tag{11}$$

and, according to (8*b*), $a = 0$ implies

$$\alpha = n\omega\sin\theta\cos\phi, \qquad \beta = n\omega\sin\theta\sin\phi, \qquad \gamma = n\omega\cos\theta. \tag{12}$$

The second solution may be written as

$$b(a+\alpha^2+\beta^2+\gamma^2) + (a-b)\gamma^2 = 0 \tag{13a}$$

and, still taking into account (8*b*), (13*a*) becomes

$$n^2m^2\omega^2 - n^2(\alpha^2+\beta^2) - m^2\gamma^2 = 0 \tag{13b}$$

satisfied with

$$\alpha = m\omega\sin\theta\cos\phi, \qquad \beta = m\omega\sin\theta\sin\phi, \qquad \gamma = n\omega\cos\theta. \tag{14}$$

According to the terminology used for uniaxial crystals [2, 4], these two types of harmonic plane waves are called *ordinary* with the wave vector (12) and *extraordinary* with (14).

Taking into account (12) and (14), the Hertz potential (6) becomes

$$\mathbf{\Pi}(\mathbf{x}) = \mathbf{\Omega}\exp[\mathrm{i}\omega u(\mathbf{x})], \qquad u(\mathbf{x}) = n(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta) \tag{15}$$

with the phase velocity $v_o^2 = 1/n^2$ and

$$\mathbf{\Pi}^\dagger(\mathbf{x}) = \mathbf{\Omega}^\dagger\exp[\mathrm{i}\omega u^\dagger(\mathbf{x})], \qquad u^\dagger(\mathbf{x}) = m(x\sin\theta\cos\phi + y\sin\theta\sin\phi) + nz\cos\theta \tag{16}$$

whose phase velocity $1/v_e^2 = m^2\sin^2\theta + n^2\cos^2\theta$ varies with the direction of propagation.

We now have to get the amplitudes $\mathbf{\Omega}$, $\mathbf{\Omega}^\dagger$ of these Hertz potentials.

Remark. Hertz–Debye potentials satisfying equation (1*b*) and discussed in appendix B generate only *ordinary* waves.

2.2. *Ordinary wave*

Taking into account (12), we get from (7*b*)

$$\begin{aligned} &\sin^2\theta\cos^2\phi\,\Omega_x + \sin^2\theta\sin\phi\cos\phi\,\Omega_y + \sin\theta\cos\theta\cos\phi\,\Omega_z = 0 \\ &\sin^2\theta\sin\phi\cos\phi\,\Omega_x + \sin^2\theta\sin^2\phi\,\Omega_y + \sin\theta\cos\theta\sin\phi\,\Omega_z = 0 \\ &n^2\sin\theta\cos\theta\cos\phi\,\Omega_x + n^2\sin\theta\cos\theta\sin\phi\,\Omega_y + (m^2 - n^2\sin^2\theta)\Omega_z = 0. \end{aligned} \tag{17}$$

The first two equations give the same relation

$$\sin\theta(\cos\phi\,\Omega_x + \sin\phi\,\Omega_y) + \cos\theta\,\Omega_z = 0 \tag{18a}$$

and the third equation is

$$n^2\sin\theta\cos\theta(\cos\phi\,\Omega_x + \sin\phi\,\Omega_y) + (m^2 - n^2\sin^2\theta)\Omega_z = 0. \tag{18b}$$

The solution of (18*a*) and (18*b*) is

$$\Omega_z = 0, \qquad \Omega_x = A\sin\phi, \qquad \Omega_y = -A\cos\phi \tag{19}$$

in which A is an arbitrary amplitude. Substituting (19) into (15) gives

$$\Pi_x = A\sin\phi\exp(\mathrm{i}\omega u), \qquad \Pi_y = -A\cos\phi\exp(\mathrm{i}\omega u), \qquad \Pi_z = 0 \tag{20}$$

from which we get at once with (3) the components of the electric and magnetic fields

$$E_x = \mu\omega^2\Pi_x, \qquad E_y = \mu\omega^2\Pi_y, \qquad E_z = 0 \tag{21a}$$

$$\begin{aligned} &H_x = n\omega^2\cos\theta\,\Pi_y, \qquad H_y = -n\omega^2\cos\theta\,\Pi_x, \\ &H_z = -n\omega^2\sin\theta(\cos\phi\,\Pi_y - \sin\phi\,\Pi_x). \end{aligned} \tag{21b}$$

Writing $u(\mathbf{x}) = \mathbf{x}\cdot\boldsymbol{\nu}$, the relations (20) imply $\mathbf{E}\cdot\boldsymbol{\nu} = 0$ and since $E_z = 0$, we also have $\mathbf{D}\cdot\boldsymbol{\nu} = 0$ where $\mathbf{D}$ is perpendicular to the plane containing the direction of propagation and the z-axis [4].

2.3. Extraordinary wave

We get from (7*b*) and (14)

$$\begin{aligned} &\sin\theta(n^2 - m^2\sin^2\phi)\Omega_x^\dagger + m^2\sin\theta\sin\phi\ \cos\ \phi\,\Omega_y^\dagger + mn\cos\theta\cos\phi\,\Omega_z^\dagger = 0 \\ &m^2\sin\theta\sin\phi\cos\phi\,\Omega_x^\dagger + \sin\theta(n^2 - m^2\cos^2\phi)\Omega_y^\dagger + mn\cos\theta\sin\phi\,\Omega_z^\dagger = 0 \\ &mn\sin\theta\cos\theta\cos\phi\,\Omega_x^\dagger + mn\sin\theta\cos\theta\sin\phi\,\Omega_y^\dagger + m^2\cos^2\theta\,\Omega_z^\dagger = 0. \end{aligned} \tag{22a}$$

Eliminating $\Omega_z^\dagger$ from the first two equations gives the simple relation

$$\sin\phi\,\Omega_x^\dagger - \cos\phi\,\Omega_y^\dagger = 0 \tag{22b}$$

and substituting (22*b*) into the second equation (22*a*) gives

$$n^2\sin\theta\,\Omega_y^\dagger + mn\cos\theta\sin\phi\,\Omega_z^\dagger = 0 \tag{22c}$$

we get from (22*b*) and (22*c*)

$$\Omega_x^\dagger = -(m/n)\cot\theta\cos\phi\,\Omega_z^\dagger, \qquad \Omega_y^\dagger = -(m/n)\cot\theta\sin\phi\,\Omega_z^\dagger. \tag{23}$$

With (23), the third equation (22*a*) is identically satisfied, so $\Omega_z^\dagger$ is arbitrary and from now on, we assume $\Omega_z^\dagger = A^\dagger$ where $A^\dagger$ is a constant amplitude.

Then, substituting (23) into (16) gives the Hertz potential

$$\{\Pi_x^\dagger, \Pi_y^\dagger, \Pi_z^\dagger\} = \{-(m/n)\cot\theta\cos\phi A^\dagger, -(m/n)\cot\theta\sin\phi A^\dagger, A^\dagger\}\exp(\mathrm{i}\omega u^\dagger) \tag{24}$$

from which we get with (3) the electric and magnetic fields

$$\{E_x^\dagger, E_y^\dagger, E_z^\dagger\} = \mu\omega^2\{\Pi_x^\dagger, \Pi_y^\dagger, \Pi_z^\dagger\}$$

$$\begin{aligned} &H_x^\dagger = -\omega^2\big(m\sin\theta\sin\phi\,\Pi_z^\dagger - n\cos\theta\,\Pi_y^\dagger\big) \\ &H_y^\dagger = -\omega^2\big(n\cos\theta\,\Pi_x^\dagger - m\sin\theta\cos\phi\,\Pi_z^\dagger\big) \end{aligned} \tag{25a}$$

$$H_z^\dagger = -\omega^2 m\sin\theta\big(\cos\phi\,\Pi_x^\dagger - \sin\phi\,\Pi_y^\dagger\big). \tag{25b}$$

For $\theta = \pi/2$, this wave reduces to a TE mode

$$E_z^\dagger = \mu\omega^2\Pi_z^\dagger, \qquad H_x^\dagger = -m\omega^2\sin\phi\,\Pi_z^\dagger, \qquad H_y^\dagger = m\omega^2\cos\phi\,\Pi_z^\dagger \tag{26}$$

propagating in the z-direction.

3. Refraction in a uniaxially anisotropic medium [6]

A TM wave coming from a homogeneous region of refractive index n_i impinges on the $z = 0$ boundary of a uniaxially anisotropic medium

$$\left\{E_x^i, E_z^i, H_y^i\right\} = \{-\cos\theta_i A_i, \sin\theta_i A_i, -n_i A_i\}\psi^i \tag{27a}$$

$$\psi^i = \exp[\mathrm{i}\omega n_i(x\sin\theta_i + z\cos\theta_i)]. \tag{27b}$$

The continuity of the exponentials at $z = 0$ implies that the fields in the uniaxial medium do not depend on y which is obtained according to (15) and (16) by imposing $\phi = 0$. Then, taking into account (20) and (21a), the ordinary wave has only two nonnull components

$$E_x = E_z = H_y = H_z = 0; \qquad E_y = -\mu\omega^2 A\psi, \qquad H_x = -n\omega^2\cos\theta A\psi \tag{28a}$$

$$\psi = \exp[\mathrm{i}\omega n(x\sin\theta + z\cos\theta)]. \tag{28b}$$

According to (24) and (25a), (25b) we get for the extraordinary wave for $\phi = 0$

$$\left\{E_x^\dagger, E_y^\dagger, E_z^\dagger\right\} = \mu\omega^2\{(m/n)A^\dagger, 0, A^\dagger\}\psi^\dagger$$

$$\left\{H_x^\dagger, H_y^\dagger, H_z^\dagger\right\} = m\omega^2\{0, (\cos^2\theta^\dagger + \sin\theta^\dagger)A^\dagger, -n^{-2}\sin\theta^\dagger\cos\theta^\dagger A^\dagger\}\psi^\dagger \tag{29a}$$

$$\psi^\dagger = \exp[\mathrm{i}\omega(mx\sin\theta^\dagger + nz\cos\theta^\dagger)]. \tag{29b}$$

To satisfy the boundary conditions on the $z = 0$ plane, we need the x, y components of the total field $\mathbf{E}^t = \mathbf{E} + \mathbf{E}^\dagger$, $\mathbf{H}^t = \mathbf{H} + \mathbf{H}^\dagger$ in the uniaxial medium and according to (28a), (29a)

$$E_x^t = E_x^\dagger, \qquad E_y^t = E_y, \qquad H_x^t = H_x, \qquad H_y^t = H_y^\dagger. \tag{30}$$

We now have to define the reflected field; taking the general solution of Maxwell's equations for harmonic plane waves with wave vector in the x, z plane gives [2]

$$\begin{aligned}\left\{E_x^r, E_z^r, H_y^r\right\} &= \{-\cos\theta_r R_1, \sin\theta_r R_1, -n_i R_1\}\psi^r \\ \left\{H_x^r, H_z^r, E_y^r\right\} &= \{-n_i\cos\theta_r R_2, \sin\theta_r R_2, R_2\}\psi^r\end{aligned} \tag{31a}$$

$$\psi^r = \exp[\mathrm{i}\omega n_i(x\sin\theta_r + z\cos\theta_r)]. \tag{31b}$$

Taking into account (27b), (28b), (29b), (31b), the continuity of the exponentials at the interface $z = 0$ supplies the Descartes–Snell relations

$$n_i\sin\theta_i = n_i\sin\theta_r = n\sin\theta = m\sin\theta^\dagger \tag{32}$$

and, the boundary conditions on the field tangential components give four equations to determine the four unknown amplitudes $R_1, R_2, A, A^\dagger$. Since $\cos\theta_r = -\cos\theta_i$, we get according to (28a), (29a), (30), (31a)

$$\begin{aligned} &E_x: \quad (R_1 - A_i)\cos\theta_i = \mu\omega^2(m/n)\cos\theta^\dagger A^\dagger \cong T_1 A^\dagger \\ &H_y: \quad n_i(R_1 + A_i) = -m\omega^2(\cos^2\theta^\dagger + \sin\theta^\dagger)A^\dagger \cong T_2 A^\dagger \\ &E_y: \quad R_2 = -\mu\omega^2 A \\ &H_x: \quad n_i\cos\theta_i R_2 = n\omega^2\cos\theta A.\end{aligned} \tag{33}$$

The last two relations imply $R_2 = A = 0$ reducing the refraction problem to a conventional one with the extraordinary wave as refracted field. A simple calculation gives from the first two equations

$$\begin{aligned} A^\dagger &= 2n_i\cos\theta_i(n_i T_1 - \cos\theta_i T_2)^{-1}A_i \\ R_1 &= A_i + 2n_i T_2(n_i T_1 - \cos\theta_i T_2)^{-1}A_i\end{aligned} \tag{34}$$

When m < n_i a total reflection happens for those angles of incidence such as $n_i/m\sin\theta_i > 1$.

So, only the *extraordinary* wave intervenes in the refraction of a TM harmonic plane wave.

Remark. An *ordinary* wave is involved in refraction if the determinant of the last two equations (33) is null which happens, taking into account (32), for an angle θ such as $\cot\theta = -\mu\cot\theta_i$,

4. Gaussian beams in uniaxial media

4.1. 2D beams

To analyse the propagation of a 2D Gaussian beam in uniaxial media, we look for the solutions of equations (5*b*) in the form

$$\mathbf{\Pi}(\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathbf{\Omega}(\alpha)\chi(\alpha), \qquad \chi(\alpha) = \exp(-\alpha^2 d^2)\exp(\mathrm{i}\alpha x + \mathrm{i}\gamma z). \tag{35}$$

Then, making $\beta = 0$ in (12) and (14) gives

$$\gamma^2 = (n^2\omega^2 - \alpha^2), \tag{36a}$$

$$\gamma^2 = (n^2/m^2)(m^2\omega^2 - \alpha^2) \tag{36b}$$

corresponding respectively to ordinary and extraordinary waves. Substituting (36*a*) into (7*b*) gives $\mathbf{\Omega}(\alpha) = 0$: *there is no ordinary 2D Gaussian beams.*

With (36*b*), we get from (7*b*)

$$\Omega_y = 0, \qquad \Omega_z = A, \qquad \Omega_x = k(\alpha)A, \qquad k(\alpha) = -(\alpha m/n)(m^2\omega^2 - \alpha^2)^{1/2} \tag{37}$$

in which A is an arbitrary constant. So, according to (35) and (37), the Hertz vector for the extra-ordinary Gaussian beam is

$$\Pi_y = 0, \qquad \Pi_z = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \chi(\alpha)A, \qquad \Pi_x = \int_{-\infty}^{\infty} \mathrm{d}\alpha\, \chi(\alpha)k(\alpha)A \tag{38a}$$

where, with $\gamma(\alpha)$ given by (36*b*)

$$\chi(\alpha) = \exp(-\alpha^2 d^2)\exp[\mathrm{i}\alpha x + \mathrm{i}\gamma(\alpha)z]. \tag{38b}$$

Then, according to (3)

$$\begin{aligned} &E_y = H_x = H_z = 0 \qquad && E_z = \mu\omega^2\Pi_z, \\ &E_x = \mu\omega^2\Pi_x, && H_y = -\omega\int_{-\infty}^{\infty} \mathrm{d}\alpha[\gamma(\alpha)k(\alpha) - \alpha]\chi(\alpha)A \end{aligned} \tag{39}$$

We may now discuss the refraction of a 2D-TM Gaussian beam, impinging from an isotropic space on the $z = 0$ boundary of a uniaxial medium [7–9].

The incident field has the representation

$$\{H_y^i, E_z^i, E_x^i\} = \int_{-\infty}^{\infty} \mathrm{d}\alpha\{1, \mathrm{i}\alpha/\omega n_i, -\mathrm{i}\gamma_i/\omega n_i\}\chi^i(\alpha)A \tag{40a}$$

$$\chi^i(\alpha) = \exp(-\alpha^2 d^2)\exp(\mathrm{i}\alpha x + \mathrm{i}\gamma_i z). \tag{40b}$$

The components of the reflected waves are

$$\{H_y^r, E_z^r, E_x^r\} = \int_{-\infty}^{\infty} \mathrm{d}\alpha\{1, \mathrm{i}\alpha/\omega n_i, \mathrm{i}\gamma_i/\omega n_i\}\chi^r(\alpha)R \tag{41a}$$

$$\chi^r(\alpha) = \exp(-\alpha^2 d^2)\exp(\mathrm{i}\alpha x - \mathrm{i}\gamma_i z). \tag{41b}$$

The continuity of the tangential components E_x, H_y at $z = 0$ supplies two relations to determine the amplitude A of the refracted wave and the reflection coefficient R:

$$A_i + R = -\omega[\gamma(\alpha)k(\alpha) - \alpha]A, \qquad \mathrm{i}\gamma_i/\omega n_i(R - A_i) = \mu\omega^2 A \tag{42}$$

with the solution

$$A = -[2/\omega p(\alpha)]A_i, \qquad R = A_i + 2\mu[\omega n_i/\gamma_i p(\alpha)]A_i \tag{43a}$$

$$p(\alpha) = \gamma(\alpha)k(\alpha) - \alpha - \mathrm{i}\mu\omega^2/\gamma_i. \tag{43b}$$

For 2D Gaussian beams as for TM harmonic plane waves, *only the extraordinary wave* intervenes in the refraction.

4.2. 3D beams

Using the polar coordinates $u = (r, \phi, z)$, equations (5*a*) for the components Π_r, Π_ϕ, Π_z of the Hertz vector become

$$\begin{aligned}(\Delta + n^2\omega^2)\Pi_r(u) &= \partial_r\nabla\cdot\mathbf{\Pi}(u)\\ (\Delta + n^2\omega^2)\Pi_\phi(u) &= 1/r\partial_\phi\nabla\cdot\mathbf{\Pi}(u)\\ (\Delta + m^2\omega^2)\Pi_z(u) &= \partial_z\nabla\cdot\mathbf{\Pi}(u), \quad \text{respectively}\end{aligned} \tag{44a}$$

in which

$$\begin{aligned}\Delta\mathbf{\Pi}(u) &= \left(\partial_r^2 + 1/r\partial_r + 1/r^2\partial_\phi^2 + \partial_z^2\right)\mathbf{\Pi}(u)\\ \nabla\cdot\mathbf{\Pi}(u) &= (\partial_r + 1/r)\Pi_r + 1/r\partial_\phi\Pi_\phi + \partial_z\Pi_z(u)\end{aligned} \tag{44b}$$

To discuss the 3D beam propagation, we use the Weber's first exponential integral [10]

$$\int_0^\infty \alpha^{\nu+1}J_\nu(\alpha r)\exp(-\alpha^2 d^2)\,\mathrm{d}\alpha = r^\nu/(2d^2)^{\nu+1}\exp(-r^2/4d^2) \tag{45}$$

in which J_ν is the Bessel function of the first kind of order ν.

So, we look for the solutions of (44*a*) with the components of the Hertz vector in the form

$$\begin{aligned}\Pi_r &= \int_0^\infty \alpha^2\,\mathrm{d}\alpha\exp(-\alpha^2 d^2)\exp(\mathrm{i}\gamma z + \mathrm{i}\phi)J_1(\alpha r)\Omega_r\\ \Pi_{\phi,z} &= \int_0^\infty \alpha\,\mathrm{d}\alpha\exp(-\alpha^2 d^2)\exp(\mathrm{i}\gamma z)J_0(\alpha r)\Omega_{\phi,z}.\end{aligned} \tag{46}$$

It is easily checked that with (46), equations (44*a*) have a solution only when two components of the Hertz vector are null. We first assume $\Pi_r = \Pi_z = 0$, so (44*a*) reduces to

$$(\Delta + n^2\omega^2)\Pi_\phi(u) = 0 \tag{47a}$$

but

$$\Delta[\exp(\mathrm{i}\gamma z)J_0(\alpha r)] = -(\alpha^2 + \gamma^2)\exp(\mathrm{i}\gamma z)J_0(\alpha r) \tag{47b}$$

so that equation (47*a*) implies $n^2\omega^2 - \alpha^2 - \gamma^2 = 0$ and according to (46)

$$\Pi_\phi = \int_0^\infty \alpha\,\mathrm{d}\alpha\exp(-\alpha^2 d^2)\exp[\mathrm{i}\gamma(\alpha)z]J_0(\alpha r)\Omega_\phi \tag{48}$$

in which Ω_ϕ is a constant and $\gamma(\alpha) = (n^2\omega^2 - \alpha^2)^{1/2}$. Then, still using (3), this Hertz potential generates the TE mode

$$\begin{aligned} &E_r = E_z = H_\phi = 0 \qquad && E_\phi = \mu\omega^2\Pi_\phi, \\ &H_r = -\mathrm{i}\omega\partial_z\Pi_\phi, && H_z = \mathrm{i}\omega(\partial_r + 1/r)\Pi_\phi. \end{aligned} \tag{49}$$

We now suppose $\Pi_\phi = \Pi_z = 0$, then (44*a*) reduces to

$$(\Delta + n^2\omega^2)\Pi_r = \partial_r(\partial + 1/r)\Pi_r. \tag{50a}$$

Now

$$\begin{aligned} &\Delta[\exp(\mathrm{i}\gamma z + \mathrm{i}\phi)J_1(\alpha r)] = -(\alpha^2 + \gamma^2)\exp(\mathrm{i}\gamma z + \mathrm{i}\phi)J_1(\alpha r) \\ &\partial_r(\partial + 1/r)J_1(\alpha r) = -\alpha^2 J_1(\alpha r). \end{aligned} \tag{50b}$$

Taking into account (50*b*), equation (50*a*) implies $n^2\omega^2 - \gamma^2 = 0$ and Π_r becomes

$$\Pi_r = \int_0^\infty \alpha^2\, \mathrm{d}\alpha \exp(-\alpha^2 d^2)\exp(\mathrm{i}n\omega z + \mathrm{i}\phi)J_1(\alpha r)\Omega_r. \tag{51}$$

Ω_r is a constant amplitude and with Π_r, we get the TE mode

$$\begin{aligned} &E_\phi = E_z = H_r = 0 \qquad && E_r = \mu\omega^2\Pi_r, \\ &H_\phi = \mathrm{i}\omega\partial_z\Pi_r, && H_z = -\mathrm{i}\omega/r\,\partial_\phi\Pi_r. \end{aligned} \tag{52}$$

Equations (44*a*) have no solution when $\Omega_r = \Omega_\phi = 0$.

For the refraction of a TE Gaussian beam, impinging from an isotropic space on the $z = 0$ boundary of a uniaxial medium, the Hertz vectors of the incident and reflected fields are

$$\begin{aligned} \Pi_\phi^i &= \int_0^\infty \alpha\, \mathrm{d}\alpha \exp(-\alpha^2 d^2)\exp(\mathrm{i}\gamma_i z)J_0(\alpha r)A_i \\ \Pi_\phi^r &= \int_0^\infty \alpha\, \mathrm{d}\alpha \exp(-\alpha^2 d^2)\exp(-\mathrm{i}\gamma_i z)J_0(\alpha r)R. \end{aligned} \tag{53}$$

Since $E_\phi = \mu\omega^2\Pi_\phi$, the continuity of E_ϕ at $z = 0$ implies according to (48) and (53)

$$A_i + R = \Omega_\phi \tag{54a}$$

and since $H_r = -\mathrm{i}\omega\partial_z\Pi_\phi$, we get from the continuity of H_r

$$\gamma_i(A_i - R) = \gamma(\alpha)\Omega_\phi. \tag{54b}$$

We deduce from (54*a*), (54*b*) in terms of the incident amplitude A_i

$$\Omega_\phi = 2\gamma_i[\gamma_i + \gamma(\alpha)]^{-1}A_i, \qquad \Omega_\phi = [\gamma_i - \gamma(\alpha)][\gamma_i + \gamma(\alpha)]^{-1}A_i. \tag{55}$$

We would have a similar result for the second TE mode.

5. Discussion

It is perhaps a bit excessive to name the Hertz vector potentials used in this work since $\mathbf{E}(\mathbf{x})$ in equation (3) has not the usual definition. Nevertheless, with these Hertz potentials, solving Maxwell's equations for harmonic fields in uniaxially anisotropic media is tantamount to solving an inhomogeneous vector Helmholtz equation. For harmonic plane waves, this last problem reduces to a set of three linear homogeneous equations and making null the determinant of this set gives two propagation directions generally obtained from the Fresnel's equation of wave normals. So, Hertz potentials may be considered as an electromagnetic approach to geometrical optics in uniaxal media [3]. But, once these directions are known, the amplitudes of ordinary and extraordinary waves are obtained as solutions of the linear homogeneous equations which make easy the analysis of refraction in a uniaxial medium.

The propagation of Gaussian beams in anisotropic media has been discussed for a long time [11]. So, it was important that the Hertz potentials $\mathbf{\Pi}$ work efficiently to tackle this problem. The Fourier representation of 2D beams, applied to $\mathbf{\Pi}$, is currently used [7], [12, where further references can be found] in electromagnetism and acoustics. It was recently shown [13] that 3D Gaussian beams may be built up by means of inhomogeneous plane waves but it seems that the Bessel representation of section 4.2 was never previously used. An interesting question is whether changing the representation of Hertz potentials modifies the propagation or not.

Nowhere in this paper, $\varepsilon, \eta, \mu, n = \sqrt{\varepsilon\mu}, m = \sqrt{\eta\mu}$ have been assumed positive so all its results hold valid in a Veselago anisotropic medium where all these parameters are negative [14]. Changing their sign requires only performing the same operation on phase velocities and field components, with the same important physical consequences as those existing in Veselago isotropic media [15, 16] due in particular to the alterations of the Descartes–Snell relations (32). But in addition the evanescent fields present in the Fourier and Bessel representations of Hertz potentials become explosive waves disturbing greatly the propagation and refraction inside a Veselago anisotropic slab [17]. The artificial realization of such materials seeming now possible, in particular with the development of nanostructures, will promote further works in this domain.

A natural question is whether Hertz potentials are an efficient tool in biaxial anisotropic media. We show in appendix A that they explicitly give the two possible propagation directions of harmonic plane waves and the corresponding field amplitudes.

Hertz–Debye potentials [4, 5, 18, 19] are an alternative technique to solve Maxwell's equations in the absence of charges. We show in appendix B that for uniaxial media, these potentials generate no extraordinary wave but in revenge, a great diversity of ordinary waves some of which, see (B.13) and (B.24), with a wave vector such as $|\mathbf{k}| = m\omega$ while with Hertz potentials $|\mathbf{k}| = n\omega$ in any case. The existence of ordinary waves with $|\boldsymbol{k}| = m\omega$ does not seem to be known. See [20, 21] for different approaches.

To sum up, problems more easily solved with the Hertz vectors used in this work than with usual methods, include: plane wave refraction on uniaxial materials, propagation of 3D-Gaussian beams in these media, plane wave propagation in biaxial anisotropic media with an immediate application to left-handed anisotropic media.

Appendix A. Hertz potentials in biaxial anisotropic media

Let the permitivity tensor be diagonal with components ε_j, $j = 1, 2, 3$. We introduce the notations

$$n_j = \sqrt{\varepsilon_j\mu}, \qquad a_j = n_j\omega - \alpha^2 - \beta^2 - \gamma^2. \tag{A.1}$$

Still considering harmonic plane wave propagation and looking for solutions of equation (4) with Hertz vectors in the form (6), we get instead of (7*b*) the equation

$$\begin{vmatrix} a_1 + \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & a_2 + \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & a_3 + \gamma^2 \end{vmatrix} \begin{vmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{vmatrix} = 0. \tag{A.2}$$

Proceeding as in (8*a*), (9) and making null the determinant of equation (A.2) gives

$$\begin{vmatrix} a_1+\alpha^2 & \alpha\beta & 0 \\ \alpha\beta & a_2+\beta^2 & -a_2\gamma/\beta \\ 0 & -a\gamma/\beta & a_3+a_2\gamma^2/\beta^2 \end{vmatrix} = 0. \tag{A.3}$$

Expanding this last determinant, we get easily

$$a_1a_2a_3 + \alpha^2 a_2a_3 + \beta^2 a_3a_1 + \gamma^2 a_1a_2 = 0 \tag{A.4}$$

that is

$$\alpha^2/a_1 + \beta^2/a_2 + \gamma^2/a_3 + 1 = 0. \tag{A.5}$$

We now write

$$a_j = n_j^2\omega^2 - b^2, \qquad b^2 = \alpha^2 + \beta^2 + \gamma^2 \tag{A.6}$$

$$\alpha = b\sin\theta\cos\phi, \qquad \beta = b\sin\theta\sin\phi, \qquad \gamma = b\cos\theta, \tag{A.7}$$

then (A.5) becomes

$$b^2\sin^2\theta\cos^2\phi\big/\left(n_1^2\omega^2 - b^2\right) + b^2\sin^2\theta\sin^2\phi\big/\left(n_2^2\omega^2 - b^2\right)$$
$$+ b^2\cos^2\theta\big/\left(n_3^2\omega^2 - b^2\right) + 1 = 0. \tag{A.8}$$

A simple calculation shows that (A.8) is a quadratic equation in b^2:

$$Ub^4 - \omega^2 Vb^2 + n_1^2n_2^2n_3^2\omega^4 = 0 \tag{A.9}$$

$$U = n_1^2 + n_2^2 + n_3^2 - \sin^2\theta\cos^2\phi\left(n_2^2 + n_3^2\right) - \sin^2\theta\sin^2\phi\left(n_3^2 + n_1^2\right) - \cos^2\theta\left(n_1^2 + n_2^2\right)$$
$$V = n_1^2n_2^2 + n_2^2n_3^2 + n_3^2n_1^2 - \sin^2\theta\cos^2\phi\, n_2^2n_3^2 - \sin^2\theta\sin^2\phi\, n_3^2n_1^2 - \cos^2\theta\, n_1^2n_2^2. \tag{A.10}$$

The solutions of (A.9) give the two propagation directions of harmonic plane waves in biaxial anisotropic media with evanescent waves when $V^2 - 4Un_1^2n_2^2n_3^2 < 0$. Once known these directions, it remains to solve (A.2) to get the field amplitudes.

Appendix B: Hertz–Debye potentials [4, 5, 18, 19]

We look for the solutions of Maxwell's equations [1*a*,*b*] in terms of two Hertz–Debye potentials satisfying equation (1*b*) and respectively called 'electric' (subscript 1) and 'magnetic' (subscript 2)

$$\begin{aligned} \mathbf{H} &= \nabla\wedge(\chi_1\mathbf{q}) + \nabla\wedge\nabla\wedge(\chi_2\mathbf{q}) \\ \mathbf{E} &= \mathbf{a}\nabla\nabla\cdot(\chi_1\mathbf{q}) - \mathrm{i}\omega\mu[\chi_1\mathbf{q} + \nabla\wedge(\chi_2\mathbf{q})] \end{aligned} \tag{B.1}$$

in which the scalar fields $\chi_{1,2}$ are solutions of a homogeneous Helmholtz equation, $\mathbf{q}$ an arbitrary vector and $\mathbf{a}$ a vector determined to satisfy equation (1*a*).

A simple calculation gives

$$\begin{aligned} [\nabla\wedge(\chi_1\mathbf{q})]_x &= q_z\partial_y\chi_1 - q_y\partial_z\chi_1 \\ [\nabla\wedge(\chi_1\mathbf{q})]_y &= q_x\partial_z\chi_1 - q_z\partial_x\chi_1 \\ [\nabla\wedge(\chi_1\mathbf{q})]_z &= q_y\partial_x\chi_1 - q_x\partial_y\chi_1 \end{aligned} \tag{B.2}$$

and

$$[\nabla \wedge \nabla(\chi_2 \mathbf{q})]_x = q_y \partial_x \partial_y \chi_2 - q_x \partial_y^2 \chi_2 - q_x \partial_z^2 \chi_2 + q_z \partial_x \partial_z \chi_2$$
$$[\nabla \wedge \nabla(\chi_2 \mathbf{q})]_y = q_z \partial_y \partial_z \chi_2 - q_y \partial_z^2 \chi_2 - q_y \partial_x^2 \chi_2 + q_x \partial_x \partial_y \chi_2 \qquad \text{(B.3)}$$
$$[\nabla \wedge \nabla(\chi_2 \mathbf{q})]_z = q_x \partial_x \partial_z \chi_2 - q_z \partial_x^2 \chi_2 - q_z \partial_y^2 \chi_2 + q_y \partial_y \partial_z \chi_2.$$

We consider successively the 'electric' (respectively 'magnetic') solutions of Maxwell's equations obtained with $\chi_2 = 0$ (respectively $\chi_1 = 0$).

A.1. 'Electric solutions' ($\chi_2 = 0$)

We get from (B.1) and (B.2) with $\nabla.(\chi_1 \mathbf{q}) = q_x \partial_x \chi_1 + q_y \partial_y \chi_1 + q_z \partial_z \chi_1$

$$H_x = q_z \partial_y \chi_1 - q_y \partial_z \chi_1$$
$$H_y = q_x \partial_z \chi_1 - q_z \partial_x \chi_1 \qquad \text{(B.4}a\text{)}$$
$$H_z = q_y \partial_x \chi_1 - q_x \partial_y \chi_1$$
$$E_x = a_x \partial_x \nabla \cdot (\chi_1 \mathbf{q}) - \mathrm{i}\omega\mu\, q_x \chi_1$$
$$E_y = a_y \partial_y \nabla \cdot (\chi_1 \mathbf{q}) - \mathrm{i}\omega\mu\, q_y \chi_1 \qquad \text{(B.4}b\text{)}$$
$$E_z = a_z \partial_z \nabla \cdot (\chi_1 \mathbf{q}) - \mathrm{i}\omega\mu\, q_z \chi_1$$

and, according to (B.4*a*)

$$(\nabla \wedge \mathbf{H})_x = q_y \delta_x \delta_y \chi_1 - q_x \delta_y^2 \chi_1 - q_x \delta_z^2 \chi_1 + q_z \delta_z \delta_x \chi_1$$
$$(\nabla \wedge \mathbf{H})_y = q_z \delta_y \delta_z \chi_1 - q_y \delta_z^2 \chi_1 - q_y \delta_x^2 \chi_1 + q_x \delta_x \delta_y \chi_1 \qquad \text{(B.5)}$$
$$(\nabla \wedge \mathbf{H})_z = q_x \delta_x \delta_z \chi_1 - q_z \delta_x^2 \chi_1 - q_z \delta_y^2 \chi_1 + q_y \delta_y \delta_z \chi_1.$$

Substituting (B.4*b*) and (B.5) into (1*a*) and taking into account (2*a*), (2*b*) gives

$$a_x = a_y = 1/\mathrm{i}\omega\varepsilon, \qquad a_z = 1/\mathrm{i}\omega\eta \qquad \text{(B.6)}$$

and the three Helmhotz equations

$$(\Delta + n^2\omega^2)\chi_1 q_x = 0, \qquad (\Delta + n^2\omega^2)\chi_1 q_y = 0, \qquad (\Delta + m^2\omega^2)\chi_1 q_z = 0 \qquad \text{(B.7)}$$

having two sets of solutions

$$(\Delta + n^2\omega^2)\chi_1 = 0 \qquad \text{with} \quad q_z = 0 \qquad \text{(B.8}a\text{)}$$
$$(\Delta + m^2\omega^2)\chi_1 = 0 \qquad \text{with} \quad q_x = q_y = 0 \qquad \text{(B.8}b\text{)}$$

A.1.1. First set of harmonic plane waves ($q_z = 0$; $|k| = n\omega$). The harmonic plane wave solutions of the Helmholtz equation (B.8*a*) are written as

$$\chi_1 = \mathrm{A}\psi_n, \qquad \psi_n = \exp[\mathrm{i}(k_x x + k_y y + k_z z)], \qquad k_x^2 + k_y^2 k_z^2 = n^2\omega^2 \qquad \text{(B.9)}$$

and we introduce the amplitudes

$$A_1 = q_x A, \qquad A_2 = q_y A, \qquad A_3 = q_z A. \qquad \text{(B.10)}$$

Then, substituting (B.9) into (B.4*b*) and taking into account (B.10) gives, since $\mathrm{q}_z = 0$,

$$H_x = -\mathrm{i}k_z A_2 \psi_n, \qquad E_x = \mathrm{i}/\omega\varepsilon\left(k_x^2 A_1 + k_x k_y A_2\right)\psi_n - \mathrm{i}\omega\mu\, A_1 \psi_n$$
$$H_y = \mathrm{i}k_z A_1 \psi_n, \qquad E_y = \mathrm{i}/\omega\varepsilon\left(k_x k_y A_1 + k_y^2 A_2\right)\psi_n - \mathrm{i}\omega\mu\, A_2 \psi_n \qquad \text{(B.11)}$$
$$H_z = \mathrm{i}(k_x A_2 - k_y A_1)\psi_n, \qquad E_z = \mathrm{i}/\omega\eta k_z (k_x A_1 + k_y A_2)\psi_n.$$

But, the amplitudes A are arbitrary and the genuine ordinary plane waves (B.11) depend only on one amplitude A : for $A_1 = k_y A$ and $A_2 = -k_x A$ we get the fields (21*a*), among other possibilities are $A_1 = k_x A$ with $A_2 = k_y A$, $A_1 = 0$, $A_2 = 0, \ldots$.

A.1.2. Second set of harmonic plane waves ($q_x = q_y = 0$; $|k| = m\omega$). The scalar field χ_1 is now a solution of the Helmholtz equation (B.8*b*)

$$\chi_1 = A\psi_m, \qquad \psi_m = \exp[\mathrm{i}(k_x x + k_y y + k_z z)], \qquad k_x^2 + k_y^2 + k_z^2 = m^2\omega^2. \tag{B.12}$$

Substituting (B.12) into (B.4*b*) and still using (B.10) gives since $q_x = q_y = 0$

$$\begin{aligned} H_x &= \mathrm{i}k_y A_z \psi_m, & E_x &= \mathrm{i}/\omega\varepsilon k_x k_z A_z \psi_m, \\ H_y &= -\mathrm{i}k_x A_z \psi_m, & E_y &= \mathrm{i}/\omega\varepsilon k_y k_z A_z \psi_m, \\ H_z &= 0, & E_z &= \mathrm{i}/\omega\varepsilon k_z^2 A_z \psi_m - \mathrm{i}\omega\mu A_z \psi_m. \end{aligned} \tag{B.13}$$

This ordinary wave with $|\mathbf{k}| = m\omega$ is a particular feature of the Hertz–Debye technique.

A.2. 'Magnetic' solutions ($\chi_1 = 0$)

We get from (B.1) and (B.3)

$$\begin{aligned} H_x &= q_y \partial_x \partial_y \chi_2 - q_x \partial_y^2 \chi_2 - q_x \partial_z^2 \chi_2 + q_z \partial_x \partial_z \chi_2, \\ H_y &= q_z \partial_y \partial_z \chi_2 - q_y \partial_z^2 \chi_2 - q_y \partial_x^2 \chi_2 + q_x \partial_x \partial_y \chi_2, \\ H_z &= q_x \partial_x \partial_z \chi_2 - q_z \partial_x^2 \chi_2 - q_x \partial_y^2 \chi_2 + q_y \partial_y \partial_z \chi_2, \end{aligned} \tag{B.14a}$$

$$\begin{aligned} E_x &= -\mathrm{i}\omega\mu(q_z \partial_y \chi_2 - q_y \partial_z \chi_2) \\ E_x &= -\mathrm{i}\omega\mu(q_x \partial_z \chi_2 - q_z \partial_x \chi_2) \\ E_x &= -\mathrm{i}\omega\mu(q_y \partial_x \chi_2 - q_x \partial_y \chi_2) \end{aligned} \tag{B.14b}$$

and according to (B.14*a*) the components of $\nabla \wedge \mathbf{H}$ are with the Laplacian operator Δ

$$\begin{aligned} (\nabla \wedge H)_x &= -q_z \partial_y \Delta \chi_2 + q_y \partial_z \Delta \chi_2 \\ (\nabla \wedge H)_y &= -q_x \partial_z \Delta \chi_2 + q_z \partial_x \Delta \chi_2 \\ (\nabla \wedge H)_z &= -q_y \partial_x \Delta \chi_2 + q_x \partial_y \Delta \chi_2. \end{aligned} \tag{B.15}$$

Substituting (B.14*b*) and (B.15) into (1.a) gives the equations

$$\begin{aligned} &(\Delta + n^2\omega^2)[q_x \partial_z \chi_2 - q_z \partial_x \chi_2] = 0 \\ &(\Delta + n^2\omega^2)[q_z \partial_y \chi_2 - q_y \partial_z \chi_2] = 0 \\ &(\Delta + m^2\omega^2)[q_y \partial_x \chi_2 - q_x \partial_y \chi_2] = 0 \end{aligned} \tag{B.16}$$

with two sets of plane wave solutions

$$(\Delta + n^2\omega^2)\chi_2 = 0 \qquad \text{with} \quad q_y \partial_x \chi_2 - q_x \partial_y \chi_2 = 0 \tag{B.17a}$$

$$(\Delta + m^2\omega^2)\chi_2 = 0 \qquad \text{with} \quad q_z = 0 \quad \text{and} \quad \partial_z \chi_2 = 0. \tag{B.17b}$$

A.2.1. First set of harmonic plane waves ($|k| = n\omega$). The scalar field χ_2 has the expression (B.9) so that the condition $q_y \partial_x \chi_2 - q_x \partial_y \chi_2 = 0$ becomes

$$k_x q_y - k_y q_x = 0. \tag{B.18}$$

Substituting (B.9) into (B.14*b*) and still using (B.11), we get

$$\begin{aligned} H_x &= \left(k_z^2 A_1 - k_x k_z A_3\right)\psi_n, & E_x &= \omega\mu(k_y A_3 - k_z A_2)\psi_n, \\ H_y &= \left(k_z^2 A_2 - k_y k_z A_3\right)\psi_n, & E_y &= \omega\mu(k_z A_1 - k_x A_3)\psi_n, \\ H_x &= \left[\left(k_x^2 + k_y^2\right)A_3 - k_z(k_x A_1 + k_y A_2)\psi_n, & E_x &= 0 \end{aligned} \tag{B.19}$$

in which according to (B.18)

$$k_x A_2 - k_y A_1 = 0 \tag{B.20}$$

a relation satisfied with $A_1 = A_2 = 0$. Then, the relations (B.19) become

$$\begin{aligned} &H_x = -k_x k_z A_3 \psi_n, && E_x = \omega\mu k_y A_3 \psi_n \\ &H_y = -k_y k_z A_3 \psi_n, && E_y = -\omega\mu k_x A_3 \psi_n \\ &H_x = \left(k_x^2 + k_y^2\right) A_3 \psi_n, && E_z = 0 \end{aligned} \tag{B.21}$$

which is the field (21*a*). The alternative $A_1 = k_x A$, $A_2 = -k_y A$ and $A_3 = 0$ gives the same expressions multiplied by $-k_z$.

A.2.2. Second set of harmonic plane waves ($|k| = m\omega$). Since $k_z = 0$ to satisfy the condition $\partial_z \chi_2 = 0$, the scalar field χ_2 becomes

$$\chi_2 = A\psi'_m, \qquad \psi'_m = \exp[\mathrm{i}(k_x x + k_y y)], \qquad k_x^2 + k_y^2 = m^2\omega^2. \tag{B.22}$$

Substituting (B.22) into (B.14*b*) gives, since $q_z = 0$,

$$\begin{aligned} &H_x = -k_y(k_x A_2 - k_y A_1)\psi'_m, && E_x = 0 \\ &H_y = k_x(k_x A_2 - k_y A_1)\psi'_m, && E_y = 0 \\ &H_z = 0, && E_x = \omega\mu(k_x A_2 - k_y A_1)\psi'_m \end{aligned} \tag{B.23}$$

which becomes with $k_x A_2 - k_y A_1 = A$ the simple TE mode with wave vector in the x-y plane

$$E_x = E_y = H_z = 0, \qquad E_z = \omega\mu A\psi'_m, \qquad H_x = -k_y A\psi'_m, \qquad H_y = k_x A\psi'_m. \tag{B.24}$$

This field such as $|\mathbf{k}| = m\omega$ is also a particular feature of the Hertz–Debye technique.

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